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INELASTIC BUCKLING OF PLATES INCLUDING SHEAR EFFECTS

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Abstract—Inelastic buckling including the effects of transverse shear is analyzed for flat rectangular plates in uniform uniaxial compression. Using a hardening material obeying the von Mises yield condition, the problem is treated according to both the incremental and deformation theories of plasticity.

Using boundary conditions which permit separation of variables it becomes possible to solve the resulting ordinary differential equations by asymptotic approximations giving closed form expressions for critical loads. The correction terms due to shear effects are obtained for three cases: (1) for infinitely long simply supported plates, (2) for square simply supported plates, and (3) for infinitely long ones simply supported on three sides and free on one unloaded edge. The analysis presented is also suitable for sandwich plates.

NOTATION

A, B, C, D	moduli in stress-strain relations
a, b, h	length, width and thickness of rectangular plate
E	Young's modulus of elasticity
$e = E/E_s - 1$	quantity used in deformation theory
E_s, E_t	secant modulus, tangent modulus
F	shear modulus in plastic range
$J_2 = \frac{1}{2}s_{ij}s_{ij}$	second invariant of deviatoric stress tensor s_{ij}
$M_{xx} = \int \sigma_{xx} z\, dz, M_{yy}, M_{xy}$	moments/unit length of the plate due to buckling
m, n	integers, number of half waves in a buckled plate in longitudinal and transverse directions respectively
$Q_x = (5/6)\int \tau_{xz}\, dz, Q_y$	transverse shear forces/unit length of the plate due to buckling
$\bar{u}_i = u, v, w$	x, y and z components of displacement vector
α	$= m\pi/a$
ϵ_{ij}	increment of strain components, elastic plus plastic, due to buckling
$\eta = \alpha y$	alternative variable
$\kappa = 1.2\sigma/F$	buckling parameter
$\lambda = E/E_t$	ratio of Young's modulus to tangent modulus
ν	Poisson's ratio
σ	buckling stress
σ_{ij}	increment of stress components due to buckling
$\phi(x, y)$	component of rotation of the normal to middle plane about y axis
$\psi(x, y)$	component of rotation of the normal to middle plane about x axis

1. INTRODUCTION

The inelastic buckling of plates using the concept of bifurcation of equilibrium and the conventional theory of plates, neglecting shear deformations, has been investigated among others by Illyushin[1], Stowell[2] and Bijlaard[3] on the basis of the constitutive relations of the J_2 deformation theory of plasticity and by Handelman and Prager[4] on the basis of the constitutive relations of the J_2 incremental theory of plasticity. Since their publications it has been well known that while the incremental theory of plasticity gives buckling loads which are sometimes absurdly higher than the experimental results, the deformation theory of plasticity. gives loads which are in reasonable agreement with the experimental results[5]. This has resulted in the apparent paradox as to why the incremental theory of plasticity, which has been accepted as a valid theory of plasticity, gives unrealistic results.

Pearson[6] improved the incremental analysis by incorporating Shanley's concept of continuous loading. However, the improvement did not significantly lower the predicted buckling stress. Subsequently, Onat and Drucker[7] analyzed the buckling of a cruciform section with thin sides and showed that if one takes into account small, unavoidable initial imperfections then the incremental theory does lead to realistic buckling loads. This explanation, however,

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leaves the buckling load somewhat dependent upon the magnitude of imperfections. The present view as expressed by Hutchinson[8] is that for a restricted range of deformations, such as met in the usual buckling problems, the J_2 deformation theory can be shown to be equivalent to a refined incremental theory of Sanders[9], taking into account the development of a corner in the yield surface under progressive compression. Thus, the bifurcation loads obtained on the basis of J_2 deformation theory are in fact those obtainable from a more complicated but physically acceptable incremental theory of plasticity[8].

The present paper analyzes inelastic buckling of plates by including transverse shear effects and using the constitutive relations of both the simple J_2 deformation and J_2 incremental theories of plasticity. While shear effects have been considered for elastic buckling[10], the extension of such analysis to inelastic buckling has yet not been carried out thoroughly.

Three cases of buckling of rectangular plates under uniform compression in longitudinal direction are considered. Bifurcation buckling stresses are obtained for (1) simply supported infinitely long plates, (2) simply supported square plates and (3) infinitely long plates supported on three sides and unsupported on one longitudinal side. Using asymptotic approximations, it becomes possible to express the buckling stresses in closed form, indicating explicitly the correction terms due to shear effects, for the above cases.

2. CONSTITUTIVE RELATIONS

The strains are considered small and the usual assumptions of isotropy and incompressibility in plastic state are made[11]. The material is considered to obey the von Mises yield condition

$$J_2 = c \tag{2.1}$$

for both the theories, where c is the hardening parameter. Loading occurs when $dJ_2 > 0$. Shanley's concept of continued loading during buckling is accepted and therefore, no unloading takes place.

Prior to buckling, the perfectly plane rectangular plate is hardening under uniform compression σ in the longitudinal direction. With buckling, the state of stress is changed by the appearance of non-zero stress increments σ_{ij} in other directions. Taking $\sigma_{33} = 0$ for the plate problems, the relations among the stress and strain increments[12] are

$$\sigma_{11} = B\epsilon_{11} + C\epsilon_{22}, \sigma_{22} = C\epsilon_{11} + D\epsilon_{22}, \sigma_{ij} = 2F\epsilon_{ij} \text{ for } i \neq j \tag{2.2}$$

where

$$B = E(\lambda + 3 + 3e)/[\lambda(5 - 4\nu + 3e) - (1 - 2\nu)^2]$$

$$C = 2E(\lambda - 1 + 2\nu)/[\lambda(5 - 4\nu + 3e) - (1 - 2\nu)^2]$$

$$D = 4E\lambda/[\lambda(5 - 4\nu + 3e) - (1 - 2\nu)^2]$$

$$F = E/(2 + 2\nu + 3e) \tag{2.3}$$

and $\lambda = E/E_t$ and $e = E/E_s$-1 are obtained from a uniaxial stress strain curve of the material of the plate.

The above relations are according to the deformation theory of plasticity[3]. The relations for the incremental theory[6] are obtained by taking $e = 0$, as noted by Bijlaard. Thus, while the analysis may be carried out on the basis of the deformation theory alone, the results for the incremental theory are obtained simply by putting $e = 0$ in the results of the deformation theory. If in addition $\lambda = 1$ is substituted, results for elastic theory are obtained.

3. GOVERNING EQUATIONS AND GENERAL SOLUTION

To derive the governing equations allowing for shear effects, a generalization of Kirchhoff's approximation is used. The changes in displacements due to buckling are taken as

$$\bar{u}_1 = u(x, y) - z\phi(x, y), \bar{u}_2 = v(x, y) - z\psi(x, y), \bar{u}_3 = w(x, y) \tag{3.1}$$

where w is the out of plane displacement and ϕ and ψ can be interpreted as the components of rotation of the normals to the middle plane $z = 0$ (Fig. 1). As is usual in the engineering theories of plates, the incompatability between the assumptions $\sigma_{33} = 0$ and $\epsilon_{33} = 0$ is ignored in the analysis.

Due to buckling, the stress components change from $-\sigma$ to $-\sigma + \sigma_{11}$ in the longitudinal direction and from 0 to σ_{ij} in the other directions, except that $\sigma_{33} = 0$. The buckled plate is in equilibrium under this changed state of stress and one may use the principle of virtual work to derive the equations of equilibrium and the necessary boundary conditions. Taking variations of (3.1) as virtual displacements and taking care to include nonlinear terms in ϵ_{11} so as not to neglect significant work terms, the following equilibrium equations for out of plane buckling[12] are obtained

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - \sigma h w_{xx} = 0 \tag{3.2}$$

subject to the boundary conditions $M_{xx} = 0$ or $\delta\phi = 0$, $M_{xy} = 0$ or $\delta\psi = 0$, $Q_x = \sigma h w_x$ or $\delta w = 0$ at the boundaries x = constant and $M_{yy} = 0$ or $\delta\psi = 0$, $M_{xy} = 0$ or $\delta\phi = 0$, $Q_y = 0$ or $\delta w = 0$ at the boundaries y = constant.

According to Shanley's concept, relations (2.2) hold throughout the thickness of the plate and the stress resultants are expressible as

$$\begin{aligned} M_{xx} &= -(h^3/12)[B\phi_x + C\psi_y], \quad M_{yy} = -(h^3/12)[C\phi_x + D\psi_y], \\ M_{xy} &= -(h^3/12)[F\phi_y + F\psi_x], \\ Q_x &= -Fkh[-w_x + \phi], \quad Q_y = -Fkh[-w_y + \psi] \end{aligned} \tag{3.3}$$

where k, a correction factor, is taken equal to 5/6, same as in the elastic case[13]. Substituting eqns (3.3) into eqns (3.2), the following governing equations, in terms of the unknown function ϕ, ψ and w, are obtained

$$\begin{aligned} (B/F)\phi_{xx} + \phi_{yy} - (10/h^2)\phi + (C/F + 1)\psi_{xy} + (10/h^2)w_x &= 0 \\ (C/F + 1)\phi_{xy} + (D/F)\psi_{yy} + \psi_{xx} - (10/h^2)\psi + (10/h^2)w_y &= 0 \\ -\phi_x - \psi_y + w_{xx} + w_{yy} - (1.2\sigma/F)w_{xx} &= 0. \end{aligned} \tag{3.4}$$

A set of boundary conditions at the loaded edges permitting separation of variables is

$$M_{xx} = w = \psi = 0 \tag{3.5}$$

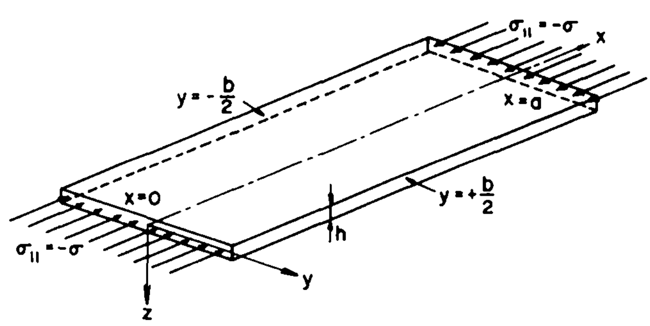

Fig. 1. Plate coordinate system.

where the third boundary condition implies $M_{xy} \neq 0$, an unlikely restraint if the edges are considered simply supported. Comments on the effect of this restraint upon the solution will be made later. A solution satisfying eqn (3.5) is

$$\phi = \Phi(y)\cos\alpha x,\ \psi = \Psi(y)\sin\alpha x,\ w = W(y)\sin\alpha x/\alpha \tag{3.6}$$

where $\alpha = m\pi/a$ and m is the number of half waves in which the plate buckles. The functions $\Phi(y)$, $\Psi(y)$ and $W(y)$ satisfy the following simultaneous ordinary differential equations

$$\begin{aligned}\Phi_{\eta\eta} - (B/F + 10/h^2\alpha^2)\Phi + (C/F + 1)\Psi_{\eta} + (10/h^2\alpha^2)W &= 0\\ -(C/F + 1)\Phi_{\eta} + D/F\Psi_{\eta\eta} - (1 + 10/h^2\alpha^2)\Psi + (10/h^2\alpha^2)W_{\eta} &= 0\\ \Phi - \Psi_{\eta} + W_{\eta\eta} + (\kappa - 1)W &= 0\end{aligned} \tag{3.7}$$

and appropriate conditions at the edges y = constant, where $\eta = \alpha y$ is a non-dimensional variable and $\kappa = 1.2\sigma/F$ is the buckling parameter. Following the usual procedure, the characteristic equation for the system of eqns (3.7) is

$$s^6 - K_1 s^4 + K_2 s^2 + K_3 = 0 \tag{3.8}$$

where

$$\begin{aligned}K_1 &= 1 + (10/h^2\alpha^2) - \kappa - R \simeq 10/h^2\alpha^2\\ K_2 &= (10/h^2\alpha^2)[4F/D + 2C/D - \kappa F/D - \kappa] - R + R\kappa + B/D\\ &\simeq (10/h^2\alpha^2)(4F/D + 2C/D)\\ K_3 &= (1 + 10/h^2\alpha^2)(B\kappa/D - B/D + 10\kappa F/Dh^2\alpha^2) \simeq (10/h^2\alpha^2)(-B/D + 10\kappa F/Dh^2\alpha^2)\end{aligned} \tag{3.9}$$

and

$$R = (-B/F + C^2/DF + 2C/D).$$

The basis of the approximations indicated in eqns (3.9) is following. Since the ratios of moduli B, C, D and F are of order unity throughout the elasto-plastic range, one may neglect κ in comparison with them. Also, considering a long simply supported plate to buckle in nearly square planes, one has $m \simeq a/b$ and therefore $10/h^2\alpha^2 \simeq b^2/h^2$ is a large number $\gg 1$. Consequently, $10\kappa F/Dh^2\alpha^2$ cannot be neglected in comparison to unity. Using these asymptotic approximations, the roots of eqn (3.8) may be expressed as

$$\begin{aligned}&s_1 = -s_2 = r,\quad r^2 \simeq 10/h^2\alpha^2;\\ &s_3 = -s_4 = p,\quad p^2 \simeq (2F + C)/D + [(2F + C)^2/D^2 - B/D + 10\kappa F/Dh^2\alpha^2]^{1/2};\\ &s_5 = -s_6 = \sqrt{(-1)}q,\ -q^2 \simeq (2F + C)/D - [(2F + C)^2/D^2 - B/D + 10\kappa F/Dh^2\alpha^2]^{1/2}.\end{aligned} \tag{3.10}$$

The general solution of eqns (3.7) is

$$\begin{aligned}\Psi &= A_1\sinh p\eta + A_2\cosh p\eta + A_3\sin q\eta\\ &\quad + A_4\cos q\eta + A_5\sinh r\eta + A_6\cosh r\eta\\ \Phi &= (A_1k_1/pk_2)\cosh p\eta + (A_2k_1/pk_2)\sinh p\eta - (A_3k_3/qk_4)\cos q\eta\end{aligned} \tag{3.11}$$

$$+ (A_4k_3/qk_4)\sin qn + (A_5k_5/rk_6)\cosh r\eta + (A_6k_5/rk_6)\sinh r\eta$$

$$W = (A_1h_1/pk_2)\cosh p\eta + (A_2h_1/pk_2)\sinh p\eta - (A_3h_2/qk_4)\cos q\eta$$

$$+ (A_4h_2/qk_4)\sin q\eta + (A_5h_3/rk_6)\cosh r\eta + (A_6h_3/rk_6)\sinh r\eta$$

where A_1 to A_6 are arbitrary constants. The expressions for h_1, k_1, etc. are obtained by substituting eqns (3.11) into eqns (3.7)

$$k_1, k_3, k_5 = (\kappa - 1 + s^2)(Ds^2/F - 1 - 10/h^2\alpha^2) + 10s^2/h^2\alpha^2$$

$$k_2, k_4, k_6 = (\kappa - 1 + s^2)(C/F + 1) + 10/h^2\alpha^2 \tag{3.12}$$

$$h_1, h_2, h_3 = (C/F + 1 - D/F)s^2 + 1 + 10/h^2\alpha^2$$

where s^2 is to be replaced respectively by p^2, $-q^2$ and r^2. The expressions corresponding to possible boundary conditions $M_{yy} = M_{xy} = Q_y = 0$ at edges y = constant are obtained by substituting eqns (3.6) and (3.11) into eqns (3.3), see [12].

4. PLATES SIMPLY SUPPORTED ON ALL FOUR SIDES

Case (a) $w = M_{yy} = \phi = 0$ *at* $y = \pm b/2$.

For a symmetrical buckling mode, satisfying the above boundary conditions, the vanishing of characteristic determinant leads to

$$\cos q\alpha b/2 = 0 \quad \text{or} \quad q = n\pi/\alpha b \tag{4.1}$$

where $n = 1, 3, 5, \ldots$ is the number of half waves in transverse direction. Since $-q^2$ is a root of eqn (3.8), following exact expression for the buckling parameter is obtained by substituting this value in eqn (3.8) and using exact expressions for K_1, K_2, K_3

$$\kappa[(n\pi/\alpha b)^4 + \{(10/h^2\alpha^2)(F/D + 1) - R\}(n\pi/\alpha b)^2 + (1 + 10/h^2\alpha^2)(B/D + 10F/Dh^2\alpha^2)]$$

$$= (n\pi/\alpha b)^6 + (-R + 1 + 10/h^2\alpha^2)(n\pi/\alpha b)^4 + \{-R + B/D + (10/h^2\alpha^2)(4F + 2C)/D\}(n\pi/\alpha b)^2$$

$$+ (1 + 10/h^2\alpha^2)B/D. \tag{4.2}$$

For minimum buckling stress $n = 1$ and m, obtained by solving $\partial\kappa/\partial\alpha = 0$, is given by

$$(\alpha b)^2 = (m\pi b/a)^2 = \pi^2\sqrt{(D/B)}[1 + \pi^2h^2/10b^2$$

$$\times \{(1 + B/F)\sqrt{(D/B)} + 2 + RD/2B + C/F + 2F/B + C/B\}] \tag{4.3}$$

up to the terms of order h^2/b^2. The minimum buckling stress is

$$\sigma = (\pi^2h^2/12b^2)[(4F + 2C) + 2\sqrt{(BD)} - (\pi^2h^2/10b^2)$$

$$\times \{(RD + 4F + 2C + BD/F)(1 + \sqrt{(D/B)} - (4F + 2C)(D/F + \sqrt{(BD)}/F) - (B + \sqrt{(BD)})D/F\}] \tag{4.4}$$

up to the terms of order h^2/b^2.

Comapring the above results with those from the conventional theory[6], it is seen that the correction terms due to shear are small, of order h^2/b^2, in both of the eqns (4.3) and (4.4). Moreover, it can be verified that one may use, in lieu of eqn (4.3), the simpler result

$$(\alpha b)^2 = (m\pi b/a)^2 = \pi^2\sqrt{(D/B)} \tag{4.5}$$

of the conventional theory and obtain the same correction terms as present in eqn (4.4). This simplification is considered permissible and is used again in case (b) to be analyzed below. For $m = n = 1$, the buckling stress for a square plate of sides $a = b$ is

$$\sigma = (\pi^2h^2/12b^2)[(4F + 2C + B + D) - (\pi^2h^2/10b^2)$$
$$\times \{2RD + 8F + 4C + 2BD/F - (4F + 2C + D)(B + D)/F\}]. \quad (4.6)$$

Case (b) $w = M_{yy} = M_{xy} = 0$ *at* $y = \pm b/2$.

This set of boundary conditions is more representative of simple support. For a symmetrical buckling mode, the characteristic determinant equated to zero gives

$$C_1 \sinh\frac{p\alpha b}{2}\cos\frac{q\alpha b}{2} + C_2\cosh\frac{p\alpha b}{2}\sin\frac{q\alpha b}{2} + C_3\cosh\frac{p\alpha b}{2}\cos\frac{q\alpha b}{2}\tanh\frac{r\alpha b}{2} = 0 \quad (4.7)$$

where C_1, C_2 and C_3 depend on the buckling stress and the material parameters and have been defined in [12]. As an alternative to numerical methods, eqn (4.7) is solved here by using an asymptotic approach, obtaining closed form expression for the buckling stress. As remarked earlier, one may take, for infinitely long plates, $(\alpha b)^2 = \pi^2\sqrt{(D/B)}$ and note that

$$r^2 \simeq 10/h^2\alpha^2 \simeq (10b^2/\pi^2h^2)\sqrt{(B/D)} \quad (4.8)$$

remains much larger than p^2 and q^2 throughout the elasto–plastic range. Furthermore, if the objective is to get an approximation for σ valid up to terms of order (h/b), one may neglect terms of order h^2/b^2 and reduce eqn (4.7) to

$$\cos\frac{q\alpha b}{2} \simeq \frac{4Fq}{Dr(p^2 + q^2)} \quad \text{or} \quad q \simeq n\pi/[\alpha b + 8F/Dr(p^2 + q^2)] \quad (4.9)$$

where $n = 1, 3, 5, \ldots$. Using $n = 1$ and $(\alpha b)^2 = \pi^2\sqrt{(D/B)}$, the expression for minimum buckling stress, with correction terms up to the order of (h/b), is

$$\sigma = (\pi^2h^2/12b^2)[4F + 2C + 2\sqrt{(BD)} - 16Fh/\sqrt{(10)}b]. \quad (4.10)$$

The buckling stress for a square plate, with $a = b$ and $\alpha b = \pi$, is

$$\sigma = (\pi^2h^2/12b^2)[4F + 2C + B + D - 16Fh/\sqrt{(10)}b]. \quad (4.11)$$

The adequacy of these asymptotic approximations has been verified by solving eqn (4.7) for several special cases using a digital computer and without making approximations.

To demonstrate the application of the above results, graphs of the buckling stress as function of b/h are shown in Fig. 2 for square plates and in Fig. 3 for infinitely long plates. The material properties used are defined by a uniaxial stress–strain curve of the form

$$\epsilon = \sigma/10{,}700 + 0.002(\sigma/61.4)^{20}; \quad \sigma \text{ in ksi}. \quad (4.12)$$

This is representative of the behaviour of 14S–T6 aluminum alloy and matches curve C of Fig. 6 of [5] up to a stress level of 62 ksi. Poisson's ratio is taken as 0.32. Results are shown for both the theories of plasticity. Graphs of the buckling stress computed by using the conventional theory are also shown.

Comparing eqn (4.10) with eqn (4.4) or eqn (4.11) with eqn (4.6), it is seen that the correction term for the case $M_{xy} = 0$ is proportional to h/b while that for the case $\phi = 0$ is proportional to h^2/b^2. This conclusion holds both for the infinitely long and the square simply supported plates. It is therefore conceivable that relaxation of the boundary condition from $\psi = 0$ to $M_{xy} = 0$ at the loaded edges could further contribute a correction term of order h/b, making the total

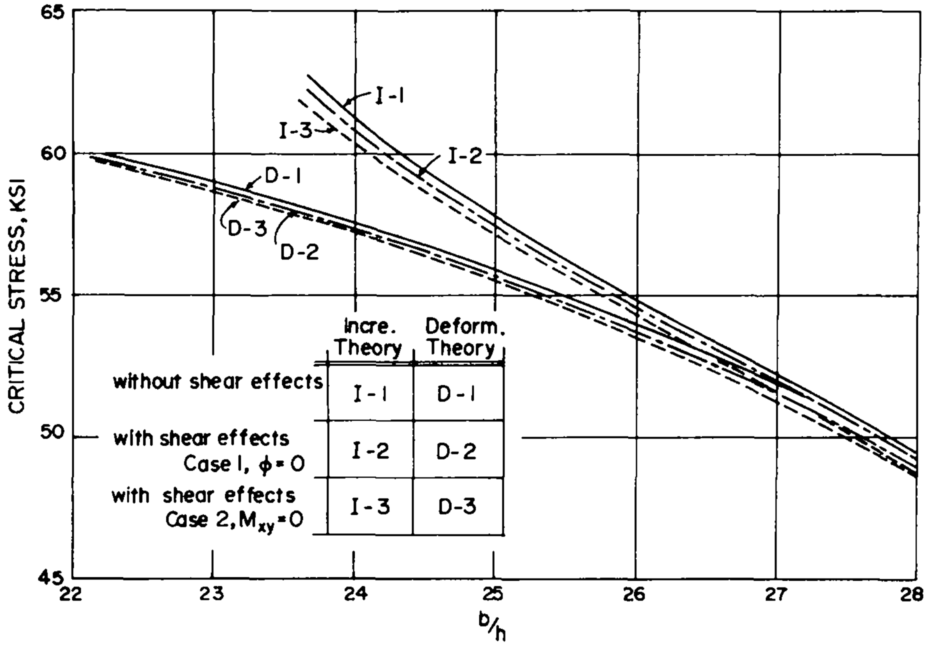

Fig. 2. Critical stress as function of b/h for square plates simply supported on all sides.

correction nearly twice as much. However, an analysis with $M_{xy} = 0$ at $x = 0$, a does not admit a product solution of the form of eqns (3.6) and some alternative approach has to be employed to solve the problem analytically.

5. BUCKLING OF LONG PLATES UNSUPPORTED ON ONE UNLOADED EDGE

The cases treated below have been identified in literature with the buckling of a column of cruciform section. This is a much discussed problem because, as is well known, the predicted buckling stress is particularly sensitive to the theory of plasticity used, incremental or deformation theory. Two cases corresponding to $\phi = 0$ and $M_{xy} = 0$ at the supported edge are treated.

Case (c) $w = M_{yy} = \phi = 0$ *at the supported edge* $y = 0$.

The boundary conditions for the unsupported edge $y = b$, are obviously $M_{yy} = M_{xy} = Q_y = 0$. The characteristic determinant leads to

$$C_1 \sinh pab \cos qab + C_2 \cosh pab \sin qab + C_3 \cosh pab \cos qab \tanh rab = 0 \tag{5.1}$$

where C_1, C_2, C_3 depend on σ and are given in [12]. It is well known from the conventional theory that the lowest buckling mode is just a single half wave and this is also assumed to be true when shear effects are included. This means that $m = 1$ and $\alpha b = \pi b/a$ is a small quantity. Therefore, since $p^2 \simeq q^2 \simeq \sqrt{(10\kappa F/Dh^2\alpha^2)}$, the trigonometric functions can be replaced by their power series expansion, say up to the terms of order $(\alpha b)^3$. Then, taking into consideration relative magnitudes of quantities, in particular the facts that $a \gg b$ and that $r^2\alpha^2 b^2 \simeq 10b^2/h^2$ is a large quantity, the buckling stress can be shown to be

$$\sigma = Fh^2/b^2(1 - h/\sqrt{(10)}b) \tag{5.2}$$

up to the correction term of order (h/b).

Case (d) $w = M_{yy} = M_{xy} = 0$ *at the supported edge* $y = 0$.

The characteristic determinant in this case is of a 6×6 matrix. Omitting the details [12], the same approximation procedure as was applied in case (c) leads to

$$\sigma = Fh^2/b^2[1 - 2h/\sqrt{(10)}b] \tag{5.3}$$

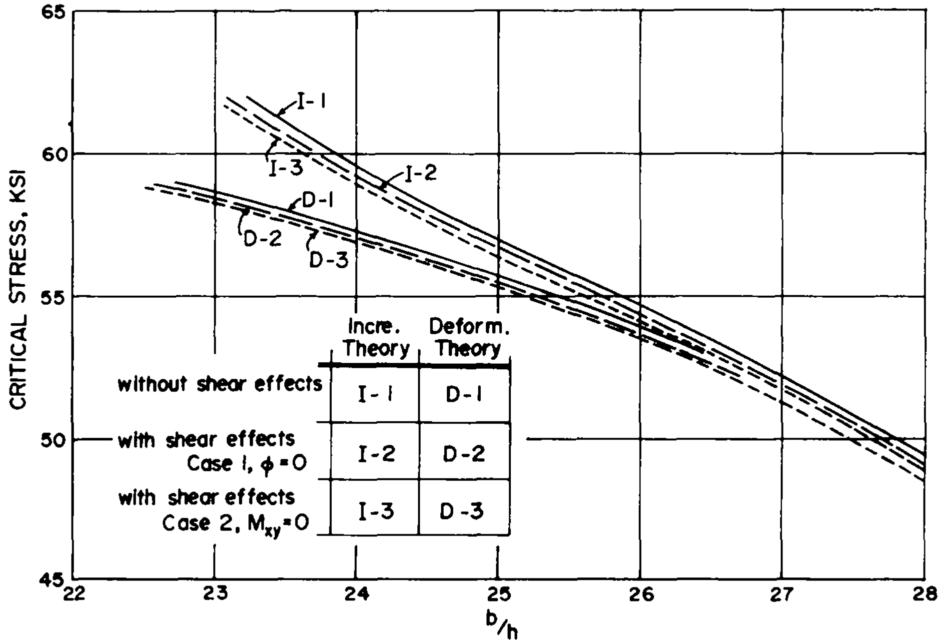

Fig. 3. Critical stress as function of b/h for infinitely long plates simply supported on all sides.

again up to the correction term of order h/b. It can be noted that changing the boundary condition from $\phi = 0$ to $M_{xy} = 0$ doubles the correction term.

Figure 4 shows numerical results. The critical stress is plotted as a function of width-to-thickness ratio for the material defined by eqn (4.12). It is seen that in the range of interest, the correction to the critical stress is 10% in the case of incremental theory and 4% in the case of deformation theory.

As a specific example, consider a long ($a/b = 10$) plate supported on three sides with $b/h = 8$. Then for this plate, the various results are

$\nu = 0.32$	Incremental Theory	Deformation Theory
without shear effects	63.5 ksi	57.4 ksi
with shear effects, $\phi = 0$	61.0 ksi	56.5 ksi
with shear effects, $M_{xy} = 0$	58.3 ksi	55.5 ksi

Thus, in this case, the improved result of the incremental theory is almost the same as the result from the deformation theory without shear effects. For lower b/h ratios the divergence between results of the two theories is, however, quite appreciable. The fact remains that F is the elastic shear modulus in case of the J_2 incremental theory and therefore the predicted buckling stress is identical to that for the elastic case, irrespective of the degree of hardening and despite the refinements due to shear effects.

6. FINAL COMMENTS

The results derived show that the correction terms due to shear effects are in general proportional to h/b, being greater for the stress-free case, $M_{xy} = 0$ than for the case of vanishing rotation, $\phi = 0$. It is noteworthy that for the buckling of a plate supported on three sides, the correction due to shear effects is generally larger (10%) for the J_2 incremental theory than that for the deformation theory (4%). The asymptotic formulas, as presented, are applicable not only to the deformation and incremental theories of plasticity but can also be specialized to the cases of elastic buckling.

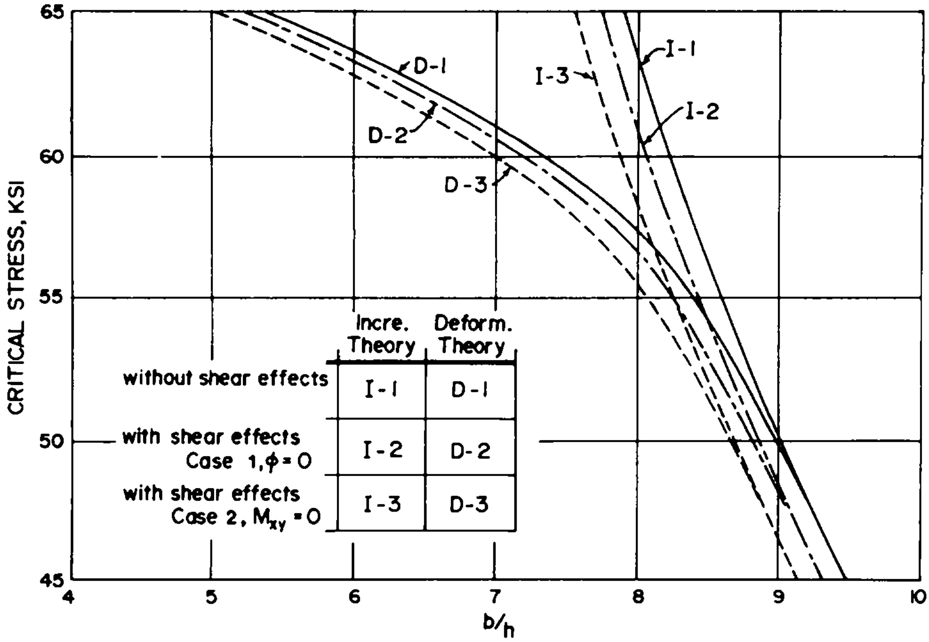

Fig. 4. Critical stress as function of b/h for infinitely long plates simply supported on three sides.

The procedure followed can be generalized to obtain governing equations which include still higher order effects, for example, the effect of transverse stress increment σ_{33} on the buckling stress. However, the resulting analysis will necessarily be more involved.

An immediately useful extension of the present analysis will be its application to the buckling of sandwich plates when the facing is in inelastic range. If the modulus of the core is low in comparison to the tangent modulus of the facing, shear effects will necessarily be much more substantial.

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